

Gordon type Theorem for measure perturbation

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August 9, 2016

Abstract

Generalizing the concept of Gordon potentials to measures we prove a version of Gordon's theorem for measures as potentials and show absence of eigenvalues for these one-dimensional Schrödinger operators.

MSC 2010: 34L05, 34L40, 81Q10

Keywords: Schrödinger operators, eigenvalue problem, quasiperiodic measure potentials

1 Introduction

According to [2], the one-dimensional Schrödinger operator $H = -\Delta + V$ has no eigenvalues if the potential $V \in L_{1,\text{loc}}(\mathbb{R})$ can be approximated by periodic potentials (in a suitable sense). The aim of this paper is to generalize this result to measures μ instead of potential functions V , i.e., to more singular potentials.

Although all statements remain valid for complex measures we only focus on real (but signed) measures μ , since we are interested in self-adjoint operators.

In the remaining part of this section we explain the situation and define the operator in question. We also describe the class of measures we are concerned with. Section 2 provides all the tools we need to prove the main theorem: $H = -\Delta + \mu$ has no eigenvalues for suitable μ . In section 3 we show some examples for Schrödinger operators with measures as potentials.

We consider a Schrödinger operator of the form

$$H = -\Delta + \mu$$

on $L_2(\mathbb{R})$. Here, $\mu = \mu_+ - \mu_-$ is a signed Borel measure on \mathbb{R} with locally finite total variation $|\mu|$.

We define H via form methods. To this end, we need to establish form boundedness of μ_- . Therefore, we restrict the class of measures we want to consider.

Definition. A signed Borel measure μ on \mathbb{R} is called *uniformly locally bounded*, if

$$\|\mu\|_{\text{loc}} := \sup_{x \in \mathbb{R}} |\mu|([x, x+1]) < \infty.$$

We call μ a *Gordon measure* if μ is uniformly locally bounded and if there exists a sequence $(\mu^m)_{m \in \mathbb{N}}$ of uniformly locally bounded periodic Borel measures with period sequence (p_m) such that $p_m \rightarrow \infty$ and for all $C \in \mathbb{R}$ we have

$$\lim_{m \rightarrow \infty} e^{Cp_m} |\mu - \mu^m|([-p_m, 2p_m]) = 0,$$

i.e., (μ^m) approximates μ on increasing intervals. Here, a Borel measure is p -periodic, if $\mu = \mu(\cdot + p)$.

Clearly, every generalized Gordon potential $V \in L_{1,\text{loc}}$ as defined in [2] induces a Gordon measure $\mu = V \lambda$, where λ is the Lebesgue measure on \mathbb{R} . Therefore, also every Gordon potential (see the original work [4]) induces a Gordon measure.

Lemma 1.1. *Let μ be a uniformly locally bounded measure. Then $|\mu|$ is $-\Delta$ -form bounded, and for all $0 < c < 1$ there is $\gamma \geq 0$ such that*

$$\int_{\mathbb{R}} |u|^2 d|\mu| \leq c \|u'\|_2^2 + \gamma \|u\|_2^2 \quad (u \in W_2^1(\mathbb{R})).$$

Proof. For $\delta \in (0, 1)$ and $n \in \mathbb{Z}$ we have

$$\|u\|_{\infty, [n\delta, (n+1)\delta]}^2 \leq 4\delta \|u'\|_{L_2(n\delta, (n+1)\delta)}^2 + \frac{4}{\delta} \|u\|_{L_2(n\delta, (n+1)\delta)}^2$$

by Sobolev's inequality.

Now, we estimate

$$\begin{aligned}
\int_{\mathbb{R}} |u|^2 d|\mu| &= \sum_{n \in \mathbb{Z}} \int_{n\delta}^{(n+1)\delta} |u|^2 d|\mu| \\
&\leq \sum_{n \in \mathbb{Z}} \|u\|_{\infty, [n\delta, (n+1)\delta]}^2 \|\mu\|_{\text{loc}} \\
&\leq \|\mu\|_{\text{loc}} \sum_{n \in \mathbb{Z}} \left(4\delta \|u'\|_{L_2(n\delta, (n+1)\delta)}^2 + \frac{4}{\delta} \|u\|_{L_2(n\delta, (n+1)\delta)}^2 \right) \\
&= 4\delta \|\mu\|_{\text{loc}} \|u'\|_2^2 + \frac{4\|\mu\|_{\text{loc}}}{\delta} \|u\|_2^2.
\end{aligned}$$

□

Let μ be a Gordon measure and define

$$\begin{aligned}
D(\tau) &:= W_2^1(\mathbb{R}), \\
\tau(u, v) &:= \int u' \bar{v}' + \int u \bar{v} d\mu.
\end{aligned}$$

Then τ is a closed symmetric semibounded form. Let H be the associated self-adjoint operator.

In [1], Ben Amor and Remling introduced a direct approach for defining the Schrödinger operator $H = -\Delta + \mu$. Since we will use some of their results we sum up the main ideas: For $u \in W_{1,\text{loc}}^1(\mathbb{R})$ define $Au \in L_{1,\text{loc}}(\mathbb{R})$ by

$$Au(x) := u'(x) - \int_0^x u(t) d\mu(t),$$

where

$$\int_0^x u(t) d\mu(t) := \begin{cases} \int_{[0,x]} u(t) d\mu(t) & \text{if } x \geq 0, \\ -\int_{(x,0)} u(t) d\mu(t) & \text{if } x < 0. \end{cases}$$

Clearly, Au is only defined as an $L_{1,\text{loc}}(\mathbb{R})$ -element. We define the operator T in $L_2(\mathbb{R})$ by

$$\begin{aligned}
D(T) &:= \{u \in L_2(\mathbb{R}); u, Au \in W_{1,\text{loc}}^1(\mathbb{R}), (Au)' \in L_2(\mathbb{R})\}, \\
Tu &:= -(Au)'.
\end{aligned}$$

Lemma 1.2. $H \subseteq T$.

Proof. Let $u \in D(H)$. Then $u \in W_2^1(\mathbb{R}) \subseteq W_{1,\text{loc}}^1(\mathbb{R})$ and $Au \in L_{1,\text{loc}}(\mathbb{R})$. Let $\varphi \in C_c^\infty(\mathbb{R}) \subseteq D(\tau)$. Using Fubini's Theorem, we compute

$$\begin{aligned}
& \int_{\mathbb{R}} (Au)(x) \varphi'(x) dx \\
&= \int_{\mathbb{R}} \left(u'(x) - \int_0^x u(t) d\mu(t) \right) \varphi'(x) dx \\
&= \int_{\mathbb{R}} u'(x) \varphi'(x) dx - \int_{\mathbb{R}} \int_0^x u(t) d\mu(t) \varphi'(x) dx \\
&= \int_{\mathbb{R}} u'(x) \varphi'(x) dx + \int_{-\infty}^0 \int_{(-\infty, t)} \varphi'(x) dx u(t) d\mu(t) - \int_0^\infty \int_{[t, \infty)} \varphi'(x) dx u(t) d\mu(t) \\
&= \int_{\mathbb{R}} u'(x) \varphi'(x) dx + \int_{-\infty}^0 u(t) \varphi(t) d\mu(t) + \int_0^\infty u(t) \varphi(t) d\mu(t) \\
&= \int_{\mathbb{R}} u' \varphi' + \int_{\mathbb{R}} u(x) \varphi(x) dx = \tau(u, \overline{\varphi}) = (Hu | \overline{\varphi}) = \int_{\mathbb{R}} Hu(x) \varphi(x) dx.
\end{aligned}$$

Hence, $(Au)' = -Hu \in L_2(\mathbb{R})$. We conclude that $Au \in W_{1,\text{loc}}^1(\mathbb{R})$ and therefore $u \in D(T)$, $Tu = -(Au)' = Hu$. \square

Remark 1.3. For $u \in D(H)$ we obtain

$$u'(x) = Au(x) + \int_0^x u(t) d\mu(t)$$

for a.a. $x \in \mathbb{R}$. Since $Au \in W_{1,\text{loc}}^1(\mathbb{R})$ and $x \mapsto \int_0^x u(t) d\mu(t)$ is continuous at all $x \in \mathbb{R}$ with $\mu(\{x\}) = 0$, u' is continuous at x for all $x \in \mathbb{R} \setminus \text{spt} \mu_p$, where μ_p is the point measure part of μ .

2 Absence of eigenvalues

We show that H has no eigenvalues. The proof is based on two observations. The first one is a stability result and will be achieved in Lemma 2.6, the

second one is an estimate of the solution for periodic measure perturbations, see Lemma 2.8.

As in [2] we start with a Gronwall Lemma, but in a more general version for locally finite measures. For the proof, see [3].

Lemma 2.1 (Gronwall). *Let μ be a locally finite Borel measure on $[0, \infty)$, $u \in \mathcal{L}_{1,\text{loc}}([0, \infty), \mu)$ and $\alpha : [0, \infty) \rightarrow [0, \infty)$ measurable. Suppose, that*

$$u(x) \leq \alpha(x) + \int_{[0,x]} u(s) d\mu(s) \quad (x \geq 0).$$

Then

$$u(x) \leq \alpha(x) + \int_{[0,x]} \alpha(s) \exp(\mu([s, x])) d\mu(s) \quad (x \geq 0).$$

For $x \in \mathbb{R}$ we abbreviate

$$I_x := [x \wedge 0, x \vee 0]$$

and

$$I_x(t) := I_x \cap ([t, x] \cup [x, t]) \quad (t \in \mathbb{R}).$$

Let μ be uniformly locally bounded. Then

$$|\mu|(I_x) \leq (|x| + 1) \|\mu\|_{\text{loc}} \quad (x \in \mathbb{R}).$$

Furthermore, if μ is periodic and locally bounded, μ is uniformly locally bounded.

Let $H := -\Delta + \mu$ and $E \in \mathbb{R}$. Then $u \in W_{1,\text{loc}}^1(\mathbb{R}) (= D(A))$ is a *solution* of $Hu = Eu$, if $-(Au)' = Eu$ in the sense of distributions (i.e., u satisfies the eigenvalue equation but without being an L_2 -function).

Lemma 2.2. *Let μ_1, μ_2 be two uniformly locally bounded measures, $E \in \mathbb{R}$ and u_1 and u_2 solutions of*

$$H_1 u_1 = E u_1, \quad H_2 u_2 = E u_2$$

subject to

$$u_1(0) = u_2(0), \quad u_1'(0+) = u_2'(0+), \quad |u_1(0)|^2 + |u_1'(0+)|^2 = 1.$$

Then there are $C_0, C \geq 0$ such that for all $x \in \mathbb{R}$

$$\begin{aligned} & \left\| \begin{pmatrix} u_1(x) \\ u_1'(x) \end{pmatrix} - \begin{pmatrix} u_2(x) \\ u_2'(x) \end{pmatrix} \right\| \\ & \leq C_0 + \int_{I_x} |u_2(t)| \, d|\mu_1 - \mu_2|(t) \\ & + C \int_{I_x} \left(C_0 + \int_{I_t} |u_2| \, d|\mu_1 - \mu_2| \right) e^{C(\lambda + |\mu_1 - E\lambda|)(I_x(t))} \, d(\lambda + |\mu_1 - E\lambda|)(t). \end{aligned}$$

Proof. Write

$$u_1(x) - u_2(x) = \int_0^x (u_1'(t) - u_2'(t)) \, dt$$

and

$$\begin{aligned} u_1'(x) - u_2'(x) &= u_1'(0+) - u_2'(0+) - (u_1(0)\mu_1(\{0\}) - u_2(0)\mu_2(\{0\})) \\ &+ \int_0^x u_1(t) \, d\mu_1(t) - \int_0^x u_2(t) \, d\mu_2(t) - \int_0^x E(u_1(t) - u_2(t)) \, dt \\ &= u_2(0) (\mu_2(\{0\}) - \mu_1(\{0\})) \\ &+ \int_0^x u_2(t) \, d(\mu_1 - \mu_2)(t) + \int_0^x (u_1(t) - u_2(t)) \, d(\mu_1 - E\lambda)(t). \end{aligned}$$

Hence,

$$\begin{aligned} \begin{pmatrix} u_1(x) - u_2(x) \\ u_1'(x) - u_2'(x) \end{pmatrix} &= \begin{pmatrix} 0 \\ u_2(0) (\mu_2(\{0\}) - \mu_1(\{0\})) \end{pmatrix} + \int_0^x \begin{pmatrix} 0 \\ u_2(t) \end{pmatrix} \, d(\mu_1 - \mu_2)(t) \\ &+ \int_0^x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1(t) - u_2(t) \\ u_1'(t) - u_2'(t) \end{pmatrix} \, d \begin{pmatrix} \lambda \\ \mu_1 - E\lambda \end{pmatrix} (t). \end{aligned}$$

We conclude, that

$$\begin{aligned}
& \left\| \begin{pmatrix} u_1(x) \\ u_1'(x) \end{pmatrix} - \begin{pmatrix} u_2(x) \\ u_2'(x) \end{pmatrix} \right\| \\
& \leq C_0 + \int_{I_x} |u_2(t)| \, d|\mu_1 - \mu_2|(t) \\
& \quad + C \int_{I_x} \left\| \begin{pmatrix} u_1(t) \\ u_1'(t) \end{pmatrix} - \begin{pmatrix} u_2(t) \\ u_2'(t) \end{pmatrix} \right\| d(\lambda + |\mu_1 - E\lambda|)(t).
\end{aligned}$$

An application of Lemma 2.1 with $\alpha(x) = C_0 + \int_{I_x} |u_2(t)| \, d|\mu_1 - \mu_2|(t)$ and $\mu = C(\lambda + |\mu_1 - E\lambda|)$ yields the assertion. \square

Remark 2.3. Regarding the proof of Lemma 2.2 we can further estimate $C_0 \leq |u_2(0)| |\mu_1 - \mu_2|(I_x)$ ($x \in \mathbb{R}$).

Lemma 2.4. *Let $E \in \mathbb{R}$ and u_0 be a solution of $-\Delta u_0 = Eu_0$. Then there is $C \geq 0$ such that $|u_0(x)| \leq Ce^{C|x|}$ for all $x \in \mathbb{R}$.*

In the following lemmas and proofs the constant C may change from line to line, but we will always state the dependence on the important quantities.

Lemma 2.5. *Let μ_1 be a locally bounded p -periodic measure, $E \in \mathbb{R}$, u_1 a solution of $H_1 u_1 = Eu_1$. Then there is $C \geq 0$ such that*

$$|u_1(x)| \leq Ce^{C|x|} \quad (x \in \mathbb{R}).$$

Proof. Let u_0 be a solution of $-\Delta u_0 = Eu_0$ subject to the same boundary conditions at 0 as u_1 . By Lemma 2.2 we have

$$\begin{aligned}
& |u_1(x) - u_0(x)| \\
& \leq C + \int_{I_x} |u_0(t)| \, d|\mu_1|(t) \\
& \quad + C \int_{I_x} \left(C + \int_{I_t} |u_0(s)| \, d|\mu_1|(s) \right) e^{C(\lambda + |\mu_1 - E\lambda|)(I_x(t))} d(\lambda + |\mu_1 - E\lambda|)(t) \\
& \leq C + |\mu_1|(I_x) Ce^{C|x|} \\
& \quad + \int_{I_x} (C + C|\mu_1|(I_t) e^{C|t|}) e^{(\lambda + |\mu_1 - E\lambda|)(I_x(t))} d(\lambda + |\mu_1 - E\lambda|)(t) \\
& \leq (C + C|\mu_1|(I_x) e^{C|x|}) (1 + e^{(\lambda + |\mu_1 - E\lambda|)(I_x)} (\lambda + |\mu_1 - E\lambda|)(I_x)).
\end{aligned}$$

Since μ_1 is periodic and locally bounded it is uniformly locally bounded and we have $|\mu_1|(I_x) \leq (|x| + 1) \|\mu_1\|_{\text{loc}}$. Furthermore, also $\mu_1 - E\lambda$ is periodic and uniformly locally bounded, so $|\mu_1 - E\lambda|(I_x) \leq (|x| + 1) \|\mu_1 - E\lambda\|_{\text{loc}}$. We conclude that

$$\begin{aligned} & |u_1(x) - u_0(x)| \\ & \leq (C + C(|x| + 1) \|\mu_1\|_{\text{loc}} e^{C|x|}) \times \\ & \quad \times (1 + e^{(|x|+1)(1+\|\mu_1-E\lambda\|_{\text{loc}})} (|x| + 1)(1 + \|\mu_1 - E\lambda\|_{\text{loc}})) \\ & \leq C e^{C|x|}, \end{aligned}$$

where C is depending on E , $\|\mu_1\|_{\text{loc}}$ and $\|\mu_1 - E\lambda\|_{\text{loc}}$. Hence,

$$|u_1(x)| \leq |u_1(x) - u_0(x)| + |u_0(x)| \leq C e^{C|x|}.$$

□

Lemma 2.6. *Let μ be a Gordon measure and (μ^m) the p_m -periodic approximants, $E \in \mathbb{R}$. Let u be a solution of $Hu = Eu$, u_m a solution of $H_m u_m = E u_m$ for $m \in \mathbb{N}$ (obeying the same boundary conditions at 0). Then there is $C \geq 0$ such that*

$$\left\| \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} - \begin{pmatrix} u_m(x) \\ u'_m(x) \end{pmatrix} \right\| \leq C e^{C|x|} |\mu - \mu^m|(I_x) \quad (x \in \mathbb{R}).$$

Proof. By Lemma 2.2 and Remark 2.3 we know that

$$\begin{aligned} & \left\| \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} - \begin{pmatrix} u_m(x) \\ u'_m(x) \end{pmatrix} \right\| \\ & \leq |u_m(0)| |\mu - \mu^m|(I_x) + \int_{I_x} |u_m(t)| d|\mu - \mu^m|(t) \\ & + C \int_{I_x} \left(|u_m(0)| |\mu - \mu^m|(I_t) + \int_{I_t} |u_m| d|\mu - \mu^m| \right) \times \\ & \quad \times e^{C(\lambda + |\mu - E\lambda|)(I_x(t))} d(\lambda + |\mu - E\lambda|)(t). \end{aligned}$$

We have

$$M := \sup_{m \in \mathbb{N}} \|\mu^m\|_{\text{loc}} < \infty,$$

since (μ^m) approximates μ . Hence, also

$$\sup_{m \in \mathbb{N}} \|\mu^m - E\lambda\|_{\text{loc}} < \infty$$

and Lemma 2.5 yields

$$|u_m(x)| \leq C e^{C|x|},$$

where C can be chosen independently of m . Therefore

$$\begin{aligned} & \left\| \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} - \begin{pmatrix} u_m(x) \\ u'_m(x) \end{pmatrix} \right\| \\ & \leq (C e^{C|x|} |\mu - \mu^m|(I_x) + C e^{C|x|} |\mu - \mu^m|(I_x)) \times \\ & \quad \times (1 + e^{C(\lambda + |\mu - E\lambda|)(I_x)} (\lambda + |\mu - E\lambda|)(I_x)). \end{aligned}$$

Since

$$|\mu - E\lambda|(I_x) \leq (|x| + 1) \|\mu - E\lambda\|_{\text{loc}},$$

we further estimate

$$\left\| \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} - \begin{pmatrix} u_m(x) \\ u'_m(x) \end{pmatrix} \right\| \leq C e^{C|x|} |\mu - \mu^m|(I_x)$$

where C is depending on $\|\mu - E\lambda\|_{\text{loc}}$ (and of course on M , $\|\mu\|_{\text{loc}}$ and E). \square

Lemma 2.6 can be regarded as a stability (or continuity) result: if the measures converge in total variation, the corresponding solutions converge as well.

Now, we focus on periodic measures and estimate the solutions. This will then be applied to the periodic approximations of our Gordon measure μ .

Remark 2.7. (a) Let f, g be two solutions of the equation $Hu = Eu$. Define their Wronskian by $W(f, g)(x) := f(x)g'(x+) - f'(x+)g(x)$. By [1], Proposition 2.5, $W(f, g)$ is constant.

(b) Let u be a solution of the equation $Hu = Eu$. Define the transfer matrix $T_E(x)$ mapping $(u(0), u'(0+))^\top$ to $(u(x), u'(x+))^\top$. Consider now the two solutions u_N, u_D subject to

$$\begin{pmatrix} u_N(0) \\ u'_N(0+) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u_D(0) \\ u'_D(0+) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then

$$T_E(x) = \begin{pmatrix} u_N(x) & u_D(x) \\ u'_N(x+) & u'_D(x+) \end{pmatrix}.$$

We obtain $\det T_E(x) = W(u_N, u_D)(x)$ and $\det T_E$ is constant, hence equals 1 for all $x \in \mathbb{R}$.

Lemma 2.8. *Let μ be p -periodic and $E \in \mathbb{R}$. Let u be a solution of $Hu = Eu$ subject to*

$$|u(0)|^2 + |u'(0+)|^2 = 1.$$

Then

$$\max \left\{ \left\| \begin{pmatrix} u(-p) \\ u'(-p+) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u(p) \\ u'(p+) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u(2p) \\ u'(2p+) \end{pmatrix} \right\| \right\} \geq \frac{1}{2}.$$

The proof of this lemma is completely analogous to the proof of [2, Lemma 2.2].

Lemma 2.9. *Let $v \in L_2 \cap BV_{\text{loc}}(\mathbb{R})$ and assume that for all $r > 0$ we have*

$$|v(x) - v(x+r)| \rightarrow 0 \quad (|x| \rightarrow \infty).$$

Then $|v(x)| \rightarrow 0$ as $|x| \rightarrow \infty$.

Proof. Without restriction, we can assume that $v \geq 0$. We prove this lemma by contradiction. Assume that $v(x) \rightarrow 0$ does not hold for $x \rightarrow \infty$. Then we can find $\delta > 0$ and (q_k) in \mathbb{R} with $q_k \rightarrow \infty$ such that $v(q_k) \geq \delta$ for all $k \in \mathbb{N}$. By square integrability of v we have $\|v\mathbb{1}_{[q_k, q_k+1]}\|_2 \rightarrow 0$. Therefore, we can find a subsequence (r_n) of (q_k) satisfying

$$\|v\mathbb{1}_{[r_n, r_n+1]}\|_2 \leq 2^{-\frac{3}{2}n} \quad (n \in \mathbb{N}).$$

Now, Chebyshev's inequality implies

$$\lambda(\{x \in [r_n, r_n+1]; v(x) \geq 2^{-n}\}) \leq 2^{2n} \|v\mathbb{1}_{[r_n, r_n+1]}\|_2^2 \leq 2^{-n} \quad (n \in \mathbb{N}).$$

Denote $A_n := \{x \in [r_n, r_n+1]; v(x) \geq 2^{-n}\} - r_n \subseteq [0, 1]$. Then $\lambda(A_n) \leq 2^{-n}$ and

$$\lambda\left(\bigcup_{n \geq 3} A_n\right) \leq \sum_{n \geq 3} \lambda(A_n) \leq 2^{-2} < 1.$$

Hence, $G := [0, 1] \setminus (\bigcup_{n \geq 3} A_n)$ has positive measure. For $r \in G$, $r > 0$ it follows

$$v(r_n + r) \leq 2^{-n} \quad (n \geq 3).$$

Therefore,

$$\liminf_{n \rightarrow \infty} |v(r_n) - v(r_n + r)| \geq \delta > 0,$$

a contradiction. \square

Lemma 2.10. *Let μ be a Gordon measure, $E \in \mathbb{R}$, $u \in D(H)$ a solution of $Hu = Eu$. Then $u(x) \rightarrow 0$ as $x \rightarrow \infty$ and $u'(x) \rightarrow 0$ as $x \rightarrow \infty$.*

Proof. Since $u \in D(H) \subseteq D(\tau) \subseteq W_2^1(\mathbb{R})$ we have $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Lemma 1.2 yields $u \in D(T)$ and $-(Au)' = Hu = Eu$. Let $r > 0$. Then, for almost all $x \in \mathbb{R}$,

$$\begin{aligned} u'(x+r) - u'(x) &= Au(x+r) - Au(x) + \int_{(x, x+r]} u(t) d\mu(t) \\ &= \int_x^{x+r} (Au)'(y) dy + \int_{(x, x+r]} u(t) d\mu(t). \end{aligned}$$

Hence,

$$\begin{aligned} |u'(x+r) - u'(x)| &\leq |E| \int_x^{x+r} |u(y)| dy + \int_{(x, x+r]} |u(t)| d|\mu|(t) \\ &\leq |E| r \|u\|_{\infty, [x, x+r]} + \|u\|_{\infty, [x, x+r]} |\mu|([x, x+r]) \\ &\leq \|u\|_{\infty, [x, x+r]} (|E| r + (r+1) \|\mu\|_{\text{loc}}). \end{aligned}$$

By Sobolev's inequality, there is $C \in \mathbb{R}$ (depending on r , but r is fixed anyway) such that

$$\|u\|_{\infty, [x, x+r]} \leq C \|u\|_{W_2^1(x, x+r)} \rightarrow 0 \quad (|x| \rightarrow \infty).$$

Thus,

$$|u'(x+r) - u'(x)| \rightarrow 0 \quad (|x| \rightarrow \infty).$$

An application of Lemma 2.9 with $v := u'$ yields $u'(x) \rightarrow 0$ as $|x| \rightarrow \infty$. \square

Now, we can state the main result of this paper.

Theorem 2.11. *Let μ be a Gordon measure. Then H has no eigenvalues.*

Proof. Let (μ^m) be the periodic approximants of μ . Let $E \in \mathbb{R}$ and u be a solution of $Hu = Eu$. Let (u_m) be the solutions for the measures (μ^m) . By Lemma 2.6 we find $m_0 \in \mathbb{N}$ such that

$$\left\| \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} - \begin{pmatrix} u_m(x) \\ u'_m(x) \end{pmatrix} \right\| \leq \frac{1}{4}$$

for $m \geq m_0$ and almost all $x \in [-p_m, 2p_m]$. By Lemma 2.8 we have

$$\limsup_{|x| \rightarrow \infty} \left(|u(x)|^2 + |u'(x)|^2 \right) \geq \frac{1}{4} > 0.$$

Hence, u cannot be in $D(H)$ by Lemma 2.10. □

3 Examples

Remark 3.1 (periodic measures). Every locally bounded periodic measure on \mathbb{R} is a Gordon measure. Thus, for $\mu := \sum_{n \in \mathbb{Z}} \delta_{n+\frac{1}{2}}$ the operator $H := -\Delta + \mu$ has no eigenvalues.

Some examples of quasi-periodic $L_{1,\text{loc}}$ -potentials can be found in [2].

For a measure μ and $x \in \mathbb{R}$ let $T_x \mu := \mu(\cdot - x)$. If μ is periodic with period p , then $T_p \mu = \mu$.

Example 3.2. Let $\alpha \in (0, 1) \setminus \mathbb{Q}$. There is a unique continued fraction expansion

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

with $a_n \in \mathbb{N}$. For $m \in \mathbb{N}$ we set $\alpha_m = \frac{p_m}{q_m}$, where

$$\begin{aligned} p_0 &= 0 & p_1 &= 1 & p_m &= a_m p_{m-1} + p_{m-2} \\ q_0 &= 1 & q_1 &= a_1 & q_m &= a_m p_{m-1} + q_{m-2}. \end{aligned}$$

α is called *Liouville number*, if there is $B \geq 0$ such that

$$|\alpha - \alpha_m| \leq B m^{-q_m}.$$

The set of Liouville numbers is a dense G_δ .

Let $\nu, \tilde{\nu}$ be 1-periodic measures and assume that there is $\gamma > 0$ such that

$$|\nu(\cdot - x) - \nu|([0, 1]) \leq |x|^\gamma \quad (x \in \mathbb{R}).$$

Define $\mu := \tilde{\nu} + \nu \circ \alpha$ and $\mu^m := \tilde{\nu} + \nu \circ \alpha_m$ for $m \in \mathbb{N}$. Then μ^m is q_m -periodic and

$$\begin{aligned} |\mu - \mu^m|([-q_m, 2q_m]) &= |\nu \circ \alpha - \nu \circ \alpha_m|([-q_m, 2q_m]) \\ &= \left| \nu \circ \frac{\alpha}{\alpha_m} - \nu \right|([-p_m, 2p_m]) \\ &\leq \sum_{n=-p_m}^{2p_m-1} \left| \nu \circ \frac{\alpha}{\alpha_m} - \nu \right|([n, n+1]). \end{aligned}$$

Now, we have

$$\begin{aligned} \left| \nu \circ \frac{\alpha}{\alpha_m} - \nu \right|([n, n+1]) &= T_{-n} \left| \nu \circ \frac{\alpha}{\alpha_m} - \nu \right|([0, 1]) \\ &= \left| T_{-n}(\nu \circ \frac{\alpha}{\alpha_m}) - T_{-n}\nu \right|([0, 1]) \\ &= \left| T_{-n}(\nu \circ \frac{\alpha}{\alpha_m}) - \nu \right|([0, 1]). \end{aligned}$$

With $g_{m,n}(y) := y + \left(\frac{\alpha}{\alpha_m} - 1\right)(y + n)$ and using periodicity of ν we obtain

$$T_{-n}(\nu \circ \frac{\alpha}{\alpha_m}) = \nu \circ g_{m,n}.$$

Hence,

$$\left| \nu \circ \frac{\alpha}{\alpha_m} - \nu \right|([n, n+1]) = |\nu \circ g_{m,n} - \nu|([0, 1]).$$

For $y \in [0, 1]$ and $n \in \{-p_m, \dots, 2p_m - 1\}$ we have

$$\left| \left(\frac{\alpha}{\alpha_m} - 1\right)(y + n) \right| \leq \left| \frac{\alpha}{\alpha_m} - 1 \right| \leq 2q_m B m^{-q_m}.$$

Thus,

$$|\nu \circ g_{m,n} - \nu|([0, 1]) \leq (2q_m B m^{-q_m})^\gamma = (2q_m B)^\gamma m^{-q_m \gamma}.$$

We conclude that

$$|\mu - \mu^m|([-q_m, 2q_m]) \leq 3p_m(2q_mB)^\gamma m^{-q_m\gamma}$$

and therefore for arbitrary $C \geq 0$

$$e^{Cq_m} |\mu - \mu^m|([-q_m, 2q_m]) \rightarrow 0 \quad (m \rightarrow \infty).$$

Hence μ is a Gordon potential and $H := -\Delta + \mu$ does not have any eigenvalues.

Acknowledgements

We want to thank Peter Stollmann for advices and hints and especially for providing a proof of Lemma 2.9. Many thanks to the unknown referee for useful comments.

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